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STOCHASTIC POROUS MEDIA EQUATIONS WITH DIVERGENCE ITÔ NOISE

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Abstract. We study the existence and uniqueness of solution to stochastic porous media equations with divergence Itô noise in infinite dimensions. The first result prove existence of a stochastic strong solution and it is essentially based on the non-local character of the noise. The second result proves existence of at least one martingale solution for the critical case corresponding to the Dirac distribution.

1. **Introduction.** We are concerned in the present work with the following stochastic porous media equation

$$\begin{cases} dX(t) - \Delta\Psi(X(t))\,dt = \sum\limits_{k=1}^{\infty} \operatorname{div}(\mu_k X(t))\, e_k d\beta_k(t), & (0,T)\times\mathcal{O}, \\ X = 0, & (0,T)\times\partial\mathcal{O}, \\ X(0) = x, & \mathcal{O}, \end{cases} \tag{1}$$

where $\mathcal{O}$ is a bounded open domain in $\mathbb{R}^d$, $d \leq 3$, with smooth boundary $\partial\mathcal{O}$ and the initial datum x is from $H^{-1}(\mathcal{O})$.

We assume that $\{e_k\}_{k\in\mathbb{N}}$ is the orthonormal basis in $L^2(\mathcal{O})$ of eigenfunctions of the homogeneous Dirichlet Laplace operator $-\Delta$. We denote by $\{\lambda_k\}_k$ the corresponding eigenvalues

$$-\Delta e_k = \lambda_k e_k, \quad k \in \mathbb{N}.$$

Through all the paper the sequence $\{\mu_k\}_{k\in\mathbb{N}}$ is assumed to be such that

$$\sum_{k=1}^{\infty} |\mu_k|^2_{\mathbb{R}^d}\, \lambda_k^2 \leq C_0 < \infty \tag{2}$$

where λ_k are the eigenvalues of the Laplace operator with homogeneous Dirichlet boundary conditions.

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The sequence $\{\beta_k\}_{k\in\mathbb{N}}$ is formed of mutually independent Brownian motions on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$ such that

$$W(t) \stackrel{def}{=} \sum_{k=1}^{\infty} e_k \beta_k(t), \quad t \geq 0,$$

is a cylindrical Wiener process in $L^2(\mathcal{O})$.

The constants $\{\mu_k\}_{k\in\mathbb{N}}$ are assumed to be from $\mathbb{R}^d$, i.e. $\mu_k = \left(\mu_k^1, \mu_k^2, ..., \mu_k^d\right)$ and the operator Ψ is maximal monotone. We recall that a function Ψ is said to be maximal monotone, i.e. $(v_1 - v_2)(u_1 - u_2) \geq 0$ for all $v_i \in \Psi(u_i)$, $i = 1, 2$, and the range $R(I + \Psi)$ of $I + \Psi$ is all $\mathbb{R}$. A standard example is $\Psi(r) = a|r|^{m-1} r - br$ where $m \geq 1$ and $a > 0$, $b \geq 0$.

Notations

We recall that $H_0^1(\mathcal{O})$ and its dual $H^{-1}(\mathcal{O})$ are the standard Sobolev spaces on $\mathcal{O}$ endowed with their usual inner products $(\cdot,\cdot)_{H_0^1(\mathcal{O})}$ and $(\cdot,\cdot)_{-1}$ and the corresponding norms $|\cdot|_{H_0^1(\mathcal{O})}$ and $|\cdot|_{-1}$ respectively. $L^m(\mathcal{O})$, $m \geq 1$, is the usual space of $m-$integrable functions endowed with the usual norm $|\cdot|_m$, and ${}_{m+1}(\cdot,\cdot)_{\frac{m+1}{m}}$ is a duality product.

For two Hilbert spaces H_1 and H_2 we denote by $L_2(H_1, H_2)$ the Hilbert-Schmidt operators from H_1 to H_2. If we have a Hilbert space H and $p, q \in [0, \infty]$, we shall denote by $L_W^q((0,T); L^p(\Omega; H))$ the space of all $q-$integrable processes $u : [0,T] \to L^p(\Omega; H)$ which are adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$. We shall denote the space of all $H-$valued adapted processes which are mean square continuous by $C_W\left([0,T]; L^2(\Omega; H)\right)$.

Through all the paper we shall denote by C a positive constant independent of the approximations, that may change in the chains of estimates.

We can rewrite equation (1) as

$$\begin{cases} dX(t) + A(X(t))\,dt = B(X(t))\,dW(t), & (0,T), \\ X(0) = x, \end{cases}$$

where

$$A : D(A) \subset H^{-1}(\mathcal{O}) \to H^{-1}(\mathcal{O})$$

is defined by

$$\begin{cases} A(u) = -\Delta\Psi(u), \quad u \in D(A) \\ D(A) = \left\{u \in H^{-1}(\mathcal{O}) \cap L^1(\mathcal{O}) : \ \Psi(u) \in H_0^1(\mathcal{O})\right\} \end{cases}$$

and

$$B : L^2(\mathcal{O}) \to L_2\left(L^2(\mathcal{O}); H^{-1}(\mathcal{O})\right)$$

is defined by

$$\begin{aligned} B(u) &: \quad L^2(\mathcal{O}) \to H^{-1}(\mathcal{O}) \\ (B(u), \varphi) &= \sum_{k=1}^{\infty} \operatorname{div}(\mu_k u)(e_k, \varphi)_2\, e_k, \quad \forall u, \varphi \in L^2(\mathcal{O}). \end{aligned}$$

Note that, for $dW(t) = \sum\limits_{i=1}^{\infty} e_i d\beta_i(t) \in L^2(\mathcal{O})$ we have that

$$B(X(t))\,dW(t) = \sum_{k=1}^{\infty} \operatorname{div}(\mu_k X(t))\, e_k d\beta_k(t).$$

Now we can easily check that B is well defined from $L^2(\mathcal{O})$ into the Hilbert-Schmidt space $L_2\left(L^2(\mathcal{O}); H^{-1}(\mathcal{O})\right)$, i.e., for any $u \in L^2(\mathcal{O})$ we have

$$\|B(u)\|^2_{L_2(L^2(\mathcal{O});H^{-1}(\mathcal{O}))} = \sum_{k=1}^{\infty} |\mathrm{div}(\mu_k u) e_k|^2_{-1} \leq C |u|_2^2, \tag{3}$$

where C is a constant.

Indeed, since $d \leq 3$, by the Sobolev embedding, it follows that

$$|e_k|_\infty \leq C |e_k|_{H^2(\mathcal{O})} \leq C |\Delta e_k|_2 \leq C\lambda_k$$

and we get by elementary computations that

$$|xe_k|^2_{-1} \leq C^2 \lambda_k^2 |x|^2_{-1}, \tag{4}$$

(see [8], [9]).

This leads to the fact that

$$\begin{aligned} \|B(u)\|^2_{L_2(L^2(\mathcal{O});H^{-1}(\mathcal{O}))} &= \sum_{k=1}^{\infty} |\langle \mu_k, \nabla u\rangle_{\mathbb{R}^d} e_k|^2_{-1} \\ &\leq C \sum_{k=1}^{\infty} \lambda_k^2 |\langle \mu_k, \nabla u\rangle_{\mathbb{R}^d}|^2_{-1}. \end{aligned} \tag{5}$$

We compute

$$\begin{aligned} |\langle \mu_k, \nabla u\rangle_{\mathbb{R}^d}|_{-1} &= \sup_{\varphi \in H_0^1(\mathcal{O}),\ |\varphi|_{H_0^1(\mathcal{O})} < 1} \left| \int_{\mathcal{O}} u(\xi) \langle \mu_k, \nabla \varphi\rangle_{\mathbb{R}^d} d\xi \right| \\ &\leq \sup_{\varphi \in H_0^1(\mathcal{O}),\ |\varphi|_{H_0^1(\mathcal{O})} < 1} |u|_2 |\langle \mu_k, \nabla \varphi\rangle_{\mathbb{R}^d}|_2 \\ &\leq \sup_{\varphi \in H_0^1(\mathcal{O}),\ |\varphi|_{H_0^1(\mathcal{O})} < 1} |u|_2 |\mu_k|_{\mathbb{R}^d} |\varphi|_{H_0^1(\mathcal{O})} \\ &\leq |u|_2 |\mu_k|_{\mathbb{R}^d}. \end{aligned}$$

Going back to (5) we get via assumption (2) that

$$\begin{aligned} \|B(u)\|^2_{L_2(L^2(\mathcal{O});H^{-1}(\mathcal{O}))} &\leq C \sum_{k=1}^{\infty} \lambda_k^2 |\mu_k|^2_{\mathbb{R}^d} |u|_2^2 \\ &\leq C |u|_2^2, \end{aligned}$$

and we obtain (3).

State of the art

We can easily see that the general existence theory mentioned below is not applicable in the present situation.

First of all, the result from [22] can not be applied in the present case since the equation is not considered in a Gelfand triple and since we don't have the assumption A_2 from [22]. Indeed, the operator B defined above is not Lipschitz from $H^{-1}(\mathcal{O})$ into $L_2\left(L^2(\mathcal{O}); H^{-1}(\mathcal{O})\right)$.

Remark 1. If we denote by $f_k = (\lambda_k)^{1/2} e_k$, $\forall k \in \mathbb{N}$, then $\{f_k\}$ is an orthonormal basis in $H^{-1}(\mathcal{O})$ and we can use it in order to define a cylindrical Wiener process in $H^{-1}(\mathcal{O})$. The operator B can then be considered from $L^2(\mathcal{O})$ to the Hilbert-Schmidt space $L_2\left(H^{-1}(\mathcal{O}); H^{-1}(\mathcal{O})\right)$. In this case it would be sufficient to have the Lipschitz property from $H^{-1}(\mathcal{O})$ into $L_2\left(H^{-1}(\mathcal{O});H^{-1}(\mathcal{O})\right)$, but this is not verified in our case neither .

Recently, the stochastic porous media equation was studied with different assumptions for the drift, with additive and multiplicative noise. See e.g. [4], [5], [6], [8], [9], [12], [24].

More precisely, the general existence theory is concerned with a stochastic porous media equation, with Itô multiplicative noise in infinite dimensions, as follows

$$\begin{cases} dX(t) - \Delta\Psi(X(t))\,dt = \sigma(X(t))\,dW(t), & (0,T)\times\mathcal{O} \\ X = 0, & (0,T)\times\partial\mathcal{O} \\ X(0) = x, & \mathcal{O} \end{cases}$$

where

$$\sigma : H^{-1}(\mathcal{O}) \to L_2\left(L^2(\mathcal{O});H^{-1}(\mathcal{O})\right)$$

is linear and Lipschitz continuous from $H^{-1}(\mathcal{O})$ into $L_2\left(L^2(\mathcal{O});H^{-1}(\mathcal{O})\right)$ and Ψ is a maximal monotone operator.

Recently in [7] the cases of $\sigma : H^{-1}(\mathcal{O}) \to L_2\left(H^{-1}(\mathcal{O});H^{-1}(\mathcal{O})\right)$ and $\sigma : L^2(\mathcal{O}) \to L_2\left(L^2(\mathcal{O});L^2(\mathcal{O})\right)$ were also studied, but they are not covering the present case.

A case of porous media equation with divergence-type noise is studied in a result from [3], but only for finite dimensions and for a Stratonovich type noise. See also [15], [19] and [25].

For different properties of the solutions of the porous media equation see [9], [13], [16], [20], [21].

With respect to the situations considered above, in the present work we assume an Itô multiplicative noise of divergence type, in infinite dimensions. To the best of our knowledge, this case was never studied before. One can also easily see that it is not covered by the previews situations since the noise is Itô-type in infinite dimensions, but not Lipschitz from $H^{-1}(\mathcal{O})$ into $L_2\left(L^2(\mathcal{O});H^{-1}(\mathcal{O})\right)$ or in the cases covered by [7].

Organization of the paper

The present paper is organized as follows.

After an introduction we have a first section which is concerned with the study of existence and uniqueness of a distributional solution for a stochastic porous media equation with non-local divergence Itô noise of the form

$$\begin{cases} dX(t) - \Delta\Psi(X(t))\,dt = \sum_{k=1}^{\infty} \operatorname{div}\left(\mu_k f * X(t)\right) e_k\, d\beta_k(t), & (0,T)\times\mathcal{O}, \\ X = 0, & (0,T)\times\partial\mathcal{O}, \\ X(0) = x, & \mathcal{O}, \end{cases}$$

where f in an $L^1(\mathcal{O})$ function.

This case can be seen as an intermediary step in the study of equation (1). In fact the function f can be seen as a regular distribution and if we take the Dirac distribution instead of f we have the singular equation (1).

The second section is concerned with the study equation (1). More precisely we shall prove the existence of at least one martingale solution of this equation.

2. **The case with non-local noise.** We are concerned in this section with the following stochastic porous media equation

$$\begin{cases} dX(t) - \Delta\Psi(X(t))\,dt = \sum_{k=1}^{\infty} \operatorname{div}(\mu_k f * X(t))\, e_k d\beta_k(t), & (0,T)\times\mathcal{O}, \\ X = 0, & (0,T)\times\partial\mathcal{O}, \\ X(0) = x, & \mathcal{O}, \end{cases} \tag{6}$$

where function f is assumed to be from $L^1(\mathcal{O})$ and $x \in H^{-1}(\mathcal{O})$.

We shall assume in this section that, in addition to (2), the following hypotheses are satisfied.

Hypotheses

i) The operator $\Psi : \mathbb{R} \to \mathbb{R}$ is a continuous, differentiable monotonically increasing function on $\mathbb{R}$, which satisfies the following conditions

$$\begin{cases} \Psi(0) = 0, \\ \Psi'(r) \le C_1 |r|^{m-1} + C_2, \quad \forall r \in \mathbb{R}, \\ j(r) = \int_0^r \Psi(s)\,ds \ge C_3 |r|^{m+1} + C_4 r^2, \quad \forall r \in \mathbb{R}, \end{cases}$$

where $C_i > 0$, $\forall i \in \{1,2,3,4\}$ and $m \ge 1$. The constant C_4 is assumed to be sufficiently large.

ii) The operator $\Psi : \mathbb{R} \to \mathbb{R}$ is strongly monotone, i.e.

$$(\Psi(r) - \Psi(s))(r-s) \ge C_5 (r-s)^2, \quad \forall r, s \in \mathbb{R},$$

where the constant $C_5 > 0$ is also assumed to be sufficiently large.

Remark 2. The assumption that C_4 and C_5 are supposed to be sufficiently large is necessary from the technical point of view to compensate the noise. The same result can be obtained if we replace this condition by C_0 sufficiently small.

As in the introduction, we can rewrite equation (6) as

$$\begin{cases} dX(t) + A(X(t))\,dt = B_f(X(t))\,dW(t), & (0,T), \\ X(0) = x, \end{cases}$$

where the operator A is defined as previously and

$$B_f : L^2(\mathcal{O}) \to L_2\left(L^2(\mathcal{O}); H^{-1}(\mathcal{O})\right)$$

is defined by

$$\begin{aligned} B_f(u) &: \quad L^2(\mathcal{O}) \to H^{-1}(\mathcal{O}) \\ (B_f(u), \varphi) &= \sum_{k=1}^{\infty} \operatorname{div}(\mu_k f * u)(e_k, \varphi)_2\, e_k, \quad \forall u, \varphi \in L^2(\mathcal{O}). \end{aligned}$$

As in the general case, we have that

$$B_f(X(t))\,dW(t) = \sum_{k=1}^{\infty} \operatorname{div}(\mu_k f * X(t))\, e_k d\beta_k(t)$$

and we can easily check that B_f is well defined from $L^2(\mathcal{O})$ into the Hilbert-Schmidt space $L_2\left(L^2(\mathcal{O}); H^{-1}(\mathcal{O})\right)$, i.e., for any $u \in L^2(\mathcal{O})$ we have

$$\|B_f(u)\|^2_{L_2(L^2(\mathcal{O});H^{-1}(\mathcal{O}))} = \sum_{k=1}^{\infty} |\operatorname{div}(\mu_k f * u)\, e_k|^2_{-1} \le C\,|u|_2^2. \tag{7}$$

Indeed, since

$$\begin{aligned} \|B_f(u)\|^2_{L_2(L^2(\mathcal{O});H^{-1}(\mathcal{O}))} &= \sum_{k=1}^{\infty} |\langle \mu_k, \nabla (f*u)\rangle_{\mathbb{R}^d} e_k|^2_{-1} \\ &\leq C\sum_{k=1}^{\infty} \lambda_k^2 |\langle \mu_k, \nabla (f*u)\rangle_{\mathbb{R}^d}|^2_{-1}, \end{aligned} \tag{8}$$

we can compute

$$\begin{aligned} |\langle \mu_k, \nabla (f*u)\rangle_{\mathbb{R}^d}|_{-1} &= \sup_{\varphi\in H_0^1(\mathcal{O}),\ |\varphi|_{H_0^1(\mathcal{O})}<1} \left| \int_{\mathcal{O}} (f*u)(\xi)\langle \mu_k, \nabla\varphi\rangle_{\mathbb{R}^d} d\xi \right| \\ &\leq \sup_{\varphi\in H_0^1(\mathcal{O}),\ |\varphi|_{H_0^1(\mathcal{O})}<1} |f*u|_2 |\langle \mu_k, \nabla\varphi\rangle_{\mathbb{R}^d}|_2 \\ &\leq \sup_{\varphi\in H_0^1(\mathcal{O}),\ |\varphi|_{H_0^1(\mathcal{O})}<1} |f*u|_2 |\mu_k|_{\mathbb{R}^d} |\varphi|_{H_0^1(\mathcal{O})} \\ &\leq |f*u|_2 |\mu_k|_{\mathbb{R}^d}. \end{aligned}$$

Keeping in mind that $f \in L^1(\mathcal{O})$ and going back to (8) we get that

$$\begin{aligned} \|B_f(u)\|^2_{L_2(L^2(\mathcal{O});H^{-1}(\mathcal{O}))} &\leq C\sum_{k=1}^{\infty} \lambda_k^2 |\mu_k|^2_{\mathbb{R}^d} |f*u|_2^2 \\ &\leq C|f|^2_{L^1(\mathcal{O})} |u|_2^2 \\ &\leq C|u|_2^2, \end{aligned}$$

and we obtained (7) where C is a constant dependent of $|f|_{L^1(\mathcal{O})}$.

We can easily see that the general existence theory mentioned before is not applicable in the present case neither. Indeed the operator B_f defined above is not Lipschitz from $H^{-1}(\mathcal{O})$ into $L_2\left(L^2(\mathcal{O}); H^{-1}(\mathcal{O})\right)$ and the result from [22] can not be applied in the present case, also since the equation is not considered in a Gelfand triple.

We shall prove now existence and uniqueness of the solution for equation (6) in the following sense.

Definition 2.1. Let $x \in H^{-1}(\mathcal{O})$. A stochastic process X which is $H^{-1}(\mathcal{O})-$valued continuous and $\mathcal{F}_t-$adapted is called a solution to equation (1) if

$$X \in L^{m+1}(\Omega \times (0,T) \times \mathcal{O})$$

for m as in assumption i) and such that

$$\begin{aligned} (X(t), e_j)_{-1} &= (x, e_j)_{-1} - \int_0^t \int_{\mathcal{O}} \Psi(X(s)) e_j d\xi ds \\ &\quad + \sum_{k=1}^{\infty} \int_0^t (\operatorname{div}(\mu_k f * X(s)) e_k, e_j)_{-1} d\beta_k(s), \end{aligned}$$

for all $j \in \mathbb{N}$, where $\{e_j\}_j$ is the orthonormal basis considered above, and for all $t \in [0,T]$.

This type of solution is inspired from [18] and [22] and was already used several times in the study of the stochastic porous media equations. See [8], [9], [14].

Note that this solution is a strong one from the stochastic point of view and a weak one from the point of view of partial differential equations.

We can now formulate the main result of this section.

Theorem 2.2. *Assume that (2) and that Hypotheses 1 hold. Then, for each* $x \in H^{-1}(\mathcal{O})$ *there is a unique solution*

$$X \in L^{m+1}(\Omega \times (0,T) \times \mathcal{O}) \cap C_W\left([0,T]; L^2\left(\Omega; H^{-1}(\mathcal{O})\right)\right)$$

to equation (1) in the sense of Definition 2.1.

Proof. **Existence of the solution**

The main idea which shall be used in this proof is the approximation of the operator B by using a mollifier, as follows.

We shall first consider a density $\rho \in C_0^\infty\left(\mathbb{R}^d\right)$ such that

$$\int_{\mathbb{R}^d} \rho(x)\, dx = 1, \quad \rho(x) = \rho(-x), \quad \rho(x) \geq 0 \quad \forall x \in \mathbb{R}^d,$$

and

$$\operatorname{supp} \rho \subset \{x \; ; \; \|x\| \leq 1\}.$$

Then, we define the function $\rho_\varepsilon(x) = \frac{1}{\varepsilon^d}\rho\left(\frac{x}{\varepsilon}\right), \varepsilon > 0$, satisfying $\rho_\varepsilon \in C_0^\infty\left(\mathbb{R}^d\right)$, $\operatorname{supp} \rho_\varepsilon \subset \{x \; \| \|x\|_{\mathbb{R}^d} \leq \varepsilon\}$, $\rho_\varepsilon(x) = \rho_\varepsilon(-x)$, $\rho_\varepsilon(x) \geq 0, \forall x \in \mathbb{R}^d$, $\int_{\mathbb{R}^d} \rho_\varepsilon(x)\, dx = 1$. Recall that such a sequence $\{\rho_\varepsilon\}_{\varepsilon > 0}$ is called a mollifier.

We can now define

$$\widetilde{f}(y) = \begin{cases} f(y), & y \in \mathcal{O} \\ 0, & y \notin \mathcal{O} \end{cases}$$

and

$$f_\varepsilon(x) = \left(\widetilde{f} * \rho_\varepsilon\right)(x) = \int_{\mathbb{R}^d} \widetilde{f}(y)\rho_\varepsilon(x-y)\, dy, \quad \forall x \in \mathbb{R}^d.$$

It is well known, by classical theory, that f_ε converges strongly in $L^1\left(\mathbb{R}^d\right)$ to $\widetilde{f}$ for $\varepsilon \to 0$ and therefore

$$|f_\varepsilon|_{L^1(\mathbb{R}^d)} \leq C\left(1 + \left|\widetilde{f}\right|_{L^1(\mathbb{R}^d)}\right) = C\left(1 + |f|_{L^1(\mathcal{O})}\right). \tag{9}$$

We shall approximate the operator B as follows:

$$\begin{aligned} B_f^\varepsilon \quad &: \quad H^{-1}(\mathcal{O}) \to L_2\left(L^2(\mathcal{O}); H^{-1}(\mathcal{O})\right) \\ B_f^\varepsilon(u) \quad &= \quad \sum_{k=1}^\infty \operatorname{div}(\mu_k f_\varepsilon * u)(e_k, \cdot)_2\, e_k, \quad \forall u \in H^{-1}(\mathcal{O}). \end{aligned}$$

We can now check that the approximating equation

$$\begin{cases} dX^\varepsilon(t) - \Delta\Psi(X^\varepsilon(t))\, dt = B_f^\varepsilon(X^\varepsilon(t))\, dW(t), & (0,T) \times \mathcal{O} \\ X^\varepsilon(t) = 0, & (0,T) \times \partial\mathcal{O} \\ X^\varepsilon(0) = x, & \mathcal{O} \end{cases} \tag{10}$$

has a unique solution

$$X^\varepsilon \in L^{m+1}(\Omega \times (0,T) \times \mathcal{O}) \cap C_W\left([0,T]; L^2\left(\Omega; H^{-1}(\mathcal{O})\right)\right)$$

in the sense of the definition above.

Since the drift satisfies already the necessary conditions, it is sufficient to check that B_f^ε is Lipschitz from $H^{-1}(\mathcal{O})$ into $L_2\left(L^2(\mathcal{O}); H^{-1}(\mathcal{O})\right)$, for each $\varepsilon > 0$ fixed.

Indeed, we have that B_f^ε is linear and also that

$$\begin{aligned}\left\|B_f^\varepsilon(u)\right\|^2_{L_2(L^2(\mathcal{O});H^{-1}(\mathcal{O}))} &= \sum_{k=1}^{\infty}\left|B_f^\varepsilon(u)e_k\right|^2_{-1} \\ &= \sum_{k=1}^{\infty}\left|\langle\mu_k,\nabla(f_\varepsilon * u)\rangle_{\mathbb{R}^d}e_k\right|^2_{-1} \\ &\leq C\sum_{k=1}^{\infty}\lambda_k^2\left|\langle\mu_k,\nabla(f_\varepsilon * u)\rangle_{\mathbb{R}^d}\right|^2_{-1}.\end{aligned} \tag{11}$$

We compute

$$\begin{aligned}&\left|\langle\mu_k,\nabla(f_\varepsilon * u)\rangle_{\mathbb{R}^d}\right|_{-1} \\ =& \sup_{\varphi\in H_0^1(\mathcal{O}),\ |\varphi|_{H_0^1(\mathcal{O})}<1}\left|\int_{\mathcal{O}}\langle\mu_k,\nabla(f_\varepsilon * u)(\xi)\rangle_{\mathbb{R}^d}\varphi(\xi)\,d\xi\right| \\ =& \sup_{\varphi\in H_0^1(\mathcal{O}),\ |\varphi|_{H_0^1(\mathcal{O})}<1}\left|\int_{\mathcal{O}}(f_\varepsilon * u)(\xi)\langle\mu_k,\nabla\varphi(\xi)\rangle_{\mathbb{R}^d}\,d\xi\right| \\ \leq& \left|f_\varepsilon * u\right|_2\left|\mu_k\right|_{\mathbb{R}^d}.\end{aligned} \tag{12}$$

Moreover we have that

$$\begin{aligned}\left|f_\varepsilon * u\right|_2 &= \left(\int_{\mathcal{O}}\left|\int_{\mathcal{O}}f_\varepsilon(x-y)u(y)\,dy\right|^2dx\right)^{1/2} \\ &\leq \left(\int_{\mathcal{O}}\left|f_\varepsilon(x-\cdot)\right|_1^2\left|u\right|^2_{-1}dx\right)^{1/2} \\ &\leq \left|u\right|_{-1}\left(\int_{\mathcal{O}}\int_{\mathcal{O}}\left|\nabla_y f_\varepsilon(x-y)\right|^2dydx\right)^{1/2} \\ &\leq C(\varepsilon)\left|u\right|_{-1}.\end{aligned} \tag{13}$$

By replacing (13) in (12) and the result in (11) we get that

$$\begin{aligned}\left\|B_f^\varepsilon(u)\right\|^2_{L_2(L^2(\mathcal{O});H^{-1}(\mathcal{O}))} &\leq C(\varepsilon)\sum_{k=1}^{\infty}\lambda_k^2\left|\mu_k\right|^2_{\mathbb{R}^d}\left|u\right|^2_{-1} \\ &\leq C(\varepsilon)\left|u\right|^2_{-1},\end{aligned}$$

by using the assumption (2). Note that the constant $C(\varepsilon)$ depends on ε and changes form line to line. We can apply Theorem 2.2 from [8] or the more recent existence result from Chapter 3 of [7], for each ε fixed.

We shall now pass to the limit in

$$\begin{aligned}\left(X^\varepsilon(t),e_j\right)_{-1} =& \left(x,e_j\right)_{-1}-\int_0^t\int_{\mathcal{O}}\Psi\left(X^\varepsilon(s)\right)e_j\,d\xi ds \\ &+\sum_{k=1}^{\infty}\int_0^t\left(\operatorname{div}\left(\mu_k f_\varepsilon * X^\varepsilon(s)\right)e_k,e_j\right)_{-1}d\beta_k(s),\quad\forall j\in\mathbb{N},\end{aligned} \tag{14}$$

for $\varepsilon\to 0$.

By using an idea similar to the one from Proposition 3.2.1 from [7] we can prove a Itô-type formula for the squared $H^{-1}(\mathcal{O})$ norm of a solution of equation (10).

More precisely, for any $j \in \mathbb{N}$ we note first that $(X^\varepsilon(t), e_j)_{-1}$ is an Itô's process and that

$$\begin{aligned} d\left(X^\varepsilon(t), e_j\right)_{-1} &= -\tfrac{m+1}{m}\left(\Psi\left(X^\varepsilon(t)\right), e_j\right)_{m+1} dt \\ &\quad + \sum_{k=1}^{\infty}\left(\operatorname{div}\left(\mu_k f_\varepsilon * X^\varepsilon(t)\right) e_k, e_j\right)_{-1} d\beta_k(t). \end{aligned}$$

Then, by applying Itô's formula as detailed in Proposition 3.2.1 from [7] and by taking the expectation, we get directly that

$$\begin{aligned} \mathbb{E}\left|X^\varepsilon(t)\right|_{-1}^2 &= |x|_{-1}^2 - 2\mathbb{E}\int_0^t \tfrac{m+1}{m}\left(\Psi\left(X^\varepsilon(s)\right), X^\varepsilon(s)\right)_{m+1} ds \\ &\quad + \mathbb{E}\int_0^t \left\|B_f^\varepsilon\left(X^\varepsilon(s)\right)\right\|_{L_2(L^2(\mathcal{O});H^{-1}(\mathcal{O}))}^2 ds. \end{aligned} \tag{15}$$

From assumption (2) we have that

$$\Psi(r)\, r \geq j(r) \geq C_3 |r|^{m+1} + C_4 r^2, \quad \forall r \in \mathbb{R},$$

which is used in the previous relation as follows

$$\begin{aligned} &\mathbb{E}\left|X^\varepsilon(t)\right|_{-1}^2 + 2\mathbb{E}\int_0^t \int_{\mathcal{O}} \left(C_3 \left|X^\varepsilon(s)\right|^{m+1} + C_4 \left|X^\varepsilon(s)\right|^2\right) d\xi ds \\ \leq\ & |x|_{-1}^2 + \mathbb{E}\int_0^t \left\|B_f^\varepsilon\left(X^\varepsilon(s)\right)\right\|_{L_2(L^2(\mathcal{O});H^{-1}(\mathcal{O}))}^2 ds. \end{aligned} \tag{16}$$

We study the last term of (16) and we get by using (4) and the assumption (2) that

$$\begin{aligned} \mathbb{E}\int_0^t \left\|B_f^\varepsilon\left(X^\varepsilon(s)\right)\right\|_{L_2(L^2(\mathcal{O});H^{-1}(\mathcal{O}))}^2 ds &= \mathbb{E}\int_0^t \sum_{k=1}^{\infty}\left|\left\langle \mu_k, \nabla\left(f_\varepsilon * X^\varepsilon(s)\right)\right\rangle_{\mathbb{R}^d} e_k\right|_{-1}^2 \\ &\leq C_6^2 \mathbb{E}\int_0^t \sum_{k=1}^{\infty} \lambda_k^2 \left|\mu_k\right|_{\mathbb{R}^d}^2 \left|\nabla\left(f_\varepsilon * X^\varepsilon(s)\right)\right|_{-1}^2 \\ &\leq C_6^2 C_0 \mathbb{E}\int_0^t \left|f_\varepsilon\right|_{L^1(\mathcal{O})}^2 \left|X^\varepsilon(s)\right|_2^2 ds \\ &\leq C_6^2 C_0 \left|f_\varepsilon\right|_{L^1(\mathcal{O})}^2 \mathbb{E}\int_0^t \left|X^\varepsilon(s)\right|_2^2 ds \\ &\leq C_6^2 C_0 C\left(1 + |f|_{L^1(\mathcal{O})}^2\right) \mathbb{E}\int_0^t \left|X^\varepsilon(s)\right|_2^2 ds \\ &\leq \widetilde{C}\mathbb{E}\int_0^t \left|X^\varepsilon(s)\right|_2^2 ds, \end{aligned}$$

where $\widetilde{C}$ is a constant independent of ε by (9).

By going back to (16) we obtain that

$$\mathbb{E}\left|X^\varepsilon(t)\right|_{-1}^2 + 2\mathbb{E}\int_0^t \int_{\mathcal{O}} \left(C_3 \left|X^\varepsilon(s)\right|^{m+1} + \left(C_4 - \widetilde{C}\right)\left|X^\varepsilon(s)\right|^2\right) d\xi ds \leq |x|_{-1}^2,$$

where C_4 is assumed to be sufficiently large and in our case this means $C_4 - \widetilde{C} > 0$.

Consequently, we can easily see that, via the hypothesis (2), we have

$$\mathbb{E}\int_0^t \int_{\mathcal{O}} \left|\Psi\left(X^\varepsilon(s)\right)\right|^{\frac{m+1}{m}} d\xi ds \leq C.$$

We obtain then the existence of

$$X \in L^{m+1}\left(\Omega \times (0,T) \times \mathcal{O}\right) \cap L^{\infty}\left(0,T;L^{2}\left(\Omega;H^{-1}\left(\mathcal{O}\right)\right)\right)$$

such that

$$\begin{aligned} X^{\varepsilon} \quad &\rightharpoonup \quad X, \quad \text{weakly in } L^{m+1}\left(\Omega \times (0,T) \times \mathcal{O}\right) \\ &\qquad \text{and weak* in } L^{\infty}\left(0,T;L^{2}\left(\Omega;H^{-1}\left(\mathcal{O}\right)\right)\right) \end{aligned}$$

and

$$\eta \in L^{\frac{m+1}{m}}\left(\Omega \times (0,T) \times \mathcal{O}\right)$$

such that

$$\Psi\left(X^{\varepsilon}\right) \rightharpoonup \eta \text{ weakly in } L^{\frac{m+1}{m}}\left(\Omega \times (0,T) \times \mathcal{O}\right).$$

We shall now study the strong convergence of the approximating solution. To this purpose we shall argue as in (15) for the difference of two approximating solutions X^{ε} and X^{λ} for $\varepsilon > 0$ and $\lambda > 0$. We get that

$$\begin{aligned} &\mathbb{E}\left|X^{\varepsilon}(t) - X^{\lambda}(t)\right|_{-1}^{2} \\ &+2\mathbb{E}\int_0^t {}_{\frac{m+1}{m}}\left(\Psi\left(X^{\varepsilon}(s)\right) - \Psi\left(X^{\lambda}(s)\right), X^{\varepsilon}(s) - X^{\lambda}(s)\right)_{m+1} ds \\ = \;& \mathbb{E}\int_0^t \left\|B_f^{\varepsilon}\left(X^{\varepsilon}(s)\right) - B_f^{\lambda}\left(X^{\lambda}(s)\right)\right\|_{L_2\left(L^2(\mathcal{O});H^{-1}(\mathcal{O})\right)}^{2} ds. \end{aligned} \tag{17}$$

Since the operator Ψ is strongly maximal monotone, we get that

$${}_{\frac{m+1}{m}}\left(\Psi\left(X^{\varepsilon}\right) - \Psi\left(X^{\lambda}\right), X^{\varepsilon} - X^{\lambda}\right)_{m+1} \geq C_5\left|X^{\varepsilon} - X^{\lambda}\right|_{2}^{2}. \tag{18}$$

Now, we only have to study the term from the right-hand side. By using the properties of the operator B_f we get that

$$\begin{aligned} &\mathbb{E}\int_0^t \left\|B_f^{\varepsilon}\left(X^{\varepsilon}(s)\right) - B_f^{\lambda}\left(X^{\lambda}(s)\right)\right\|_{L_2\left(L^2(\mathcal{O});H^{-1}(\mathcal{O})\right)}^{2} ds \\ \leq \;& C\mathbb{E}\int_0^t \left|\left(f_{\varepsilon} * X^{\varepsilon}\right)(s) - \left(f_{\lambda} * X^{\lambda}\right)(s)\right|_{2}^{2} ds \\ \leq \;& C\mathbb{E}\int_0^t \left|\left(f_{\varepsilon} * X^{\varepsilon}\right)(s) - \left(f_{\varepsilon} * X^{\lambda}\right)(s)\right|_{2}^{2} ds \\ &\qquad + C\mathbb{E}\int_0^t \left|\left(f_{\varepsilon} * X^{\lambda}\right)(s) - \left(f_{\lambda} * X^{\lambda}\right)(s)\right|_{2}^{2} ds \\ \leq \;& \overline{C}\mathbb{E}\int_0^t \left|f_{\varepsilon}\right|_{L^1(\mathcal{O})}^{2}\left|X^{\varepsilon}(s) - X^{\lambda}(s)\right|_{2}^{2} ds \\ &\qquad + C\mathbb{E}\int_0^t \left|f_{\varepsilon} - f_{\lambda}\right|_{L^1(\mathcal{O})}^{2}\left|X^{\lambda}(s)\right|_{2}^{2} ds, \end{aligned} \tag{19}$$

where the constants $\overline{C}$ and C are independents of ε.

We shall replace now (18) and (19) in (17) and, since the constant C_5 is also assumed to be sufficiently large, we get that

$$\mathbb{E}\left|X^{\varepsilon}(t) - X^{\lambda}(t)\right|_{-1}^{2}$$

$$
\begin{aligned}
& +\left(2C_5-\overline{C}\left(1+|f|^2_{L^1(\mathcal{O})}\right)\right)\mathbb{E}\int_0^t \left|X^{\varepsilon}(s)-X^{\lambda}(s)\right|_2^2 ds \qquad (20)\\
\leq \;& C\left|f_{\varepsilon}-f_{\lambda}\right|^2_{L^1(\mathcal{O})}\mathbb{E}\int_0^t \left|X^{\lambda}(s)\right|_2^2 ds,
\end{aligned}
$$

where $2C_5-\overline{C}\left(1+|f|^2_{L^1(\mathcal{O})}\right)$ is a positive constant independent of ε and λ.

Finally, since $\{f_{\varepsilon}\}_{\varepsilon}$ is strongly converging to f in $L^1(\mathcal{O})$ and $\mathbb{E}\int_0^t \left|X^{\lambda}(s)\right|_2^2 ds$ is bounded uniformly in λ, we get that

$$
\begin{aligned}
X^{\varepsilon} \quad \to \quad & X, \quad \text{strongly in } L^2(\Omega\times(0,T)\times\mathcal{O})\\
& \text{and strongly in } C\left([0,T];L^2\left(\Omega;H^{-1}(\mathcal{O})\right)\right).
\end{aligned}
$$

In order to pass to the limit in (14) we still have to study what happens in the last term of this relation.

More precisely, keeping in mind that

$$
\sum_{k=1}^{\infty}\int_0^t \left(\operatorname{div}\left(\mu_k f * X(s)\right)e_k, e_j\right)_{-1} d\beta_k(s)
$$

is well defined, we can first see that

$$
\begin{aligned}
& \sum_{k=1}^{\infty}\int_0^t \left(\left(\operatorname{div}\left(\mu_k f_{\varepsilon} * X^{\varepsilon}(s)\right)e_k, e_j\right)_{-1}-\left(\operatorname{div}\left(\mu_k f * X(s)\right)e_k, e_j\right)_{-1}\right) d\beta_k(s)\\
= \;& \sum_{k=1}^{\infty}\int_0^t \left(\left\langle \mu_k, \nabla\left(f_{\varepsilon} * X^{\varepsilon}-f * X\right)\right\rangle_{\mathbb{R}^d} e_k, e_j\right)_{-1} d\beta_k(s).
\end{aligned}
$$

Since

$$
\sum_{k=1}^{\infty}\int_0^t \left(e_j, \operatorname{div}\left(\mu_k f * X(s)\right)e_k\right)_{-1} d\beta_k(s)=\int_0^t \left(e_j, B(X(s))\,dW(s)\right)_{-1}
$$

we get, by using the Itô isometry for stochastic integrals with cylindrical Wiener processes, that

$$
\begin{aligned}
& \mathbb{E}\left(\int_0^t \left(e_j, B_f^{\varepsilon}\left(X^{\varepsilon}(s)\right)-B_f\left(X(s)\right)dW(s)\right)_{-1}\right)^2 \qquad (21)\\
= \;& \sum_{k=1}^{\infty}\mathbb{E}\left(\int_0^t \left(\left\langle \mu_k, \nabla\left(f_{\varepsilon} * X^{\varepsilon}-f * X\right)e_k\right\rangle_{\mathbb{R}^d}, e_j\right)_{-1} d\beta_k(s)\right)^2.
\end{aligned}
$$

See e.g. [17], Proposition 2.3.5 from [23] or Remark 6.3.2 from [7].

On the other hand we compute

$$
\begin{aligned}
& \mathbb{E}\left(\int_0^t \left(\left\langle \mu_k, \nabla\left(f_{\varepsilon} * X^{\varepsilon}-f * X\right)\right\rangle_{\mathbb{R}^d} e_k, e_j\right)_{-1} d\beta_k(s)\right)^2\\
= \;& \mathbb{E}\int_0^t \left(\left\langle \mu_k, \nabla\left(f_{\varepsilon} * X^{\varepsilon}-f * X\right)\right\rangle_{\mathbb{R}^d} e_k, e_j\right)_{-1}^2 ds\\
\leq \;& \mathbb{E}\int_0^t \mu_k^2\lambda_k^2\lambda_j^2 \left|\left|\nabla\left(f_{\varepsilon} * X^{\varepsilon}-f * X\right)\right|_{\mathbb{R}^d}\right|_{-1}^2 ds\\
\leq \;& \mathbb{E}\int_0^t \mu_k^2\lambda_k^2\lambda_j^2 \left|\left(f_{\varepsilon} * X^{\varepsilon}-f * X\right)\right|_2^2 ds
\end{aligned}
$$

$$
\begin{aligned}
&\leq \quad 2\mu_k^2\lambda_k^2\lambda_j^2\mathbb{E}\int_0^t \left(\left|(f_\varepsilon * X^\varepsilon - f * X^\varepsilon)\right|_2^2 + \left|(f * X^\varepsilon - f * X)\right|_2^2\right) ds \\
&\leq \quad 2\mu_k^2\lambda_k^2\lambda_j^2 \left(\left|f_\varepsilon - f\right|_{L^1(\mathcal{O})}^2 \mathbb{E}\int_0^t \left|X^\varepsilon\right|_2^2 ds + \left|f\right|_{L^1(\mathcal{O})}^2 \mathbb{E}\int_0^t \left|X^\varepsilon - X\right|_2^2 ds\right).
\end{aligned}
$$

Going back to (21) and replacing the computation above, we get that

$$
\begin{aligned}
&\mathbb{E}\left(\int_0^t \left(e_j, B_f^\varepsilon\left(X^\varepsilon\left(s\right)\right) - B_f\left(X\left(s\right)\right) dW\left(s\right)\right)_{-1}\right)^2 \\
\leq \quad & 2\sum_{k=1}^\infty \mu_k^2\lambda_k^2\lambda_j^2 \left(\mathbb{E}\int_0^t \left|X^\varepsilon\right|_2^2 ds + \left|f\right|_{L^1(\mathcal{O})}^2 \mathbb{E}\int_0^t \left|X^\varepsilon - X\right|_2^2 ds\right) \\
\leq \quad & C\left(\left|f_\varepsilon - f\right|_{L^1(\mathcal{O})}^2 + \mathbb{E}\int_0^t \left|X^\varepsilon - X\right|_2^2 ds\right)
\end{aligned}
$$

which goes to zero for $\varepsilon \to 0$.

We finally obtain that, on a subsequence, we have that

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\sum_{k=1}^\infty \int_0^t \left(\operatorname{div}\left(\mu_k f_\varepsilon * X^\varepsilon\left(s\right)\right) e_k, e_j\right)_{-1} d\beta_k\left(s\right) \\
= \quad & \sum_{k=1}^\infty \int_0^t \left(\operatorname{div}\left(\mu_k f * X\left(s\right)\right) e_k, e_j\right)_{-1} d\beta_k\left(s\right).
\end{aligned}
$$

We can now pass to the limit in (14) and get that

$$
\begin{aligned}
\left(X\left(t\right), e_j\right)_{-1} \quad = \quad & \left(x, e_j\right)_{-1} - \int_0^t \int_{\mathcal{O}} \eta\left(s\right) e_j d\xi ds \\
& + \sum_{k=1}^\infty \int_0^t \left(\operatorname{div}\left(\mu_k f * X\left(s\right)\right) e_k, e_j\right)_{-1} d\beta_k\left(s\right), \quad \forall j \in \mathbb{N}.
\end{aligned} \tag{22}
$$

In order to finish the proof we only need to show that $\eta = \Psi\left(X\right)$ a.e. on $\Omega \times (0,T) \times \mathcal{O}$. Since the operator $X \longmapsto \Psi\left(X\right)$ is maximal monotone in the duality pair $L^{m+1}\left(\Omega \times (0,T) \times \mathcal{O}\right)$ et $L^{\frac{m+1}{m}}\left(\Omega \times (0,T) \times \mathcal{O}\right)$ it is sufficient to show that

$$
\liminf_{\lambda\to 0}\mathbb{E}\int_0^t \int_{\mathcal{O}} \Psi\left(X^\varepsilon\left(s\right)\right) X^\varepsilon\left(s\right) d\xi ds \leq \mathbb{E}\int_0^t \int_{\mathcal{O}} \eta\left(s\right) X\left(s\right) d\xi ds. \tag{23}
$$

To prove (23) we shall use the same argument as in [9].

We first note that

$$
\begin{aligned}
\mathbb{E}\left|X^\varepsilon\left(t\right)\right|_{-1}^2 \quad = \quad & \left|x\right|_{-1}^2 - 2\mathbb{E}\int_0^t {}_{\frac{m+1}{m}}\left(\Psi\left(X^\varepsilon\left(s\right)\right), X^\varepsilon\left(s\right)\right)_{m+1} ds \\
& + \mathbb{E}\int_0^t \left\|B_f^\varepsilon\left(X^\varepsilon\left(s\right)\right)\right\|_{L_2(L^2(\mathcal{O});H^{-1}(\mathcal{O}))}^2 ds.
\end{aligned}
$$

Computing as in (19) in the last term of the previous relation and using that $X^\varepsilon \to X$ strongly in $C\left([0,T]; L^2\left(\Omega; H^{-1}\left(\mathcal{O}\right)\right)\right)$ we get that

$$
\begin{aligned}
&\liminf_{\lambda\to 0}\mathbb{E}\int_0^t \int_{\mathcal{O}} \Psi\left(X^\varepsilon\left(s\right)\right) X^\varepsilon\left(s\right) d\xi ds + \frac{1}{2}\mathbb{E}\left|X\left(t\right)\right|_{-1}^2 \\
\leq \quad & \frac{1}{2}\left|x\right|_{-1}^2 + \frac{1}{2}\mathbb{E}\int_0^t \left\|B_f\left(X\left(s\right)\right)\right\|_{L_2(L^2(\mathcal{O});H^{-1}(\mathcal{O}))}^2 ds.
\end{aligned} \tag{24}
$$

On the other hand, via Itô's formula applied to (22) and summation over j, we obtain that

$$\begin{aligned} & \mathbb{E}\int_0^t \int_{\mathcal{O}} \eta(s) X(s)\, d\xi ds + \frac{1}{2}\mathbb{E}\left|X(t)\right|_{-1}^2 \\ \leq\ & \frac{1}{2}\left|x\right|_{-1}^2 + \frac{1}{2}\mathbb{E}\int_0^t \left\|B_f(X(s))\right\|^2_{L_2(L^2(\mathcal{O});H^{-1}(\mathcal{O}))} ds. \end{aligned} \tag{25}$$

Combining (24) and (25) we get (23) and this completes the proof of the existence part.

Uniqueness of the solution

Concerning the uniqueness of the solution, it is sufficient to take two solutions $X^{(1)}$ and $X^{(2)}$ with the same starting point, and, by repeating the argument above, we obtain

$$\begin{aligned} & \mathbb{E}\left|X^{(1)}(t) - X^{(2)}(t)\right|_{-1}^2 \\ & +2\mathbb{E}\int_0^t {}_{\frac{m+1}{m}}\left(\Psi\left(X^{(1)}(s)\right) - \Psi\left(X^{(2)}(s)\right), X^{(1)}(s) - X^{(2)}(s)\right)_{m+1} ds \\ =\ & \mathbb{E}\int_0^t \left\|B_f\left(X^{(1)}(s)\right) - B_f\left(X^{(2)}(s)\right)\right\|^2_{L_2(L^2(\mathcal{O});H^{-1}(\mathcal{O}))} ds \\ \leq\ & C\mathbb{E}\int_0^t \left|f * \left(X^{(1)}(s) - X^{(2)}(s)\right)\right|_2^2 ds. \end{aligned} \tag{26}$$

Since Ψ is strongly monotone and the constant C_5 is assumed to be sufficiently large, we obtain that $X^{(1)} = X^{(2)}$ and the proof is complete. □

3. The critical case.

In this section we shall prove existence of at least one martingale solution for equation (1).

Hypothesis

i) The operator $\Psi : \mathbb{R} \to \mathbb{R}$ is a C^1, monotonically increasing function on $\mathbb{R}$, which satisfies the following conditions

$$\begin{cases} \Psi(0) = 0, \\ \Psi'(r) \leq C_1 |r|^{m-1} + C_2, \quad \forall r \in \mathbb{R}, \\ C_6 |r|^{m+1} + C_7 r^2 \geq j(r) = \int_0^r \Psi(s)\, ds \geq C_3 |r|^{m+1} + C_4 r^2, \quad \forall r \in \mathbb{R}, \end{cases}$$

where $C_i > 0$, $\forall i \in \{1, 2, 3, 4, 6, 7\}$ and $m > 1$. The constant C_4 is assumed to be sufficiently large.

ii) The operator $\Psi : \mathbb{R} \to \mathbb{R}$ is strongly monotone, i.e.

$$(\Psi(r) - \Psi(s))(r - s) \geq C_5 (r - s)^2, \quad \forall r, s \in \mathbb{R},$$

where the constant $C_5 > 0$ is also assumed to be sufficiently large.

Remark 3. The case $m = 1$ corresponding to a Lipschitz operator Ψ can be obtained by a natural adaptation of the same computations. More precisely the condition $m > 1$ is essential in the application of the Egorov theorem, for the bound of $\left\{\widetilde{X}^{\varepsilon}\right\}_{\varepsilon}$ in $L^{m+1}\left(\widetilde{\Omega} \times (0,T) \times \mathcal{O}\right)$. The same bound can be obtained by applying the Itô formula with the Lyapunov function $\varphi(r) = |r|_p^p$ for $p > 2$. (see e.g. Lemme 3.1 from [9]).

Definition 3.1. Let $x \in H^{-1}(\mathcal{O})$. We call weak martingale solution to equation (1) a tuple $\left(\widetilde{\widetilde{\Omega}}, \widetilde{\widetilde{\mathcal{F}}}, \left(\widetilde{\widetilde{\mathcal{F}_t}}\right)_{t\geq 0}, \widetilde{\widetilde{\mathbb{P}}}, \widetilde{\widetilde{W}}, \widetilde{\widetilde{X}}\right)$ where $\left(\widetilde{\widetilde{\Omega}}, \widetilde{\widetilde{\mathcal{F}}}, \left(\widetilde{\widetilde{\mathcal{F}_t}}\right)_{t\geq 0}, \widetilde{\widetilde{\mathbb{P}}}\right)$ is a filtrated probability space where there are defined a $\left(\widetilde{\widetilde{\mathcal{F}_t}}\right)_{t\geq 0}$ – Wiener process $\widetilde{\widetilde{W}}$ and a continuous $\left(\widetilde{\widetilde{\mathcal{F}_t}}\right)_{t\geq 0}$ – adapted, $H^{-1}(\mathcal{O})$ – valued process $\widetilde{\widetilde{X}}$ such that

$$\begin{aligned}\left(\widetilde{\widetilde{X}}(t), e_j\right)_{-1} &= (x, e_j)_{-1} - \int_0^t \int_{\mathcal{O}} \Psi\left(\widetilde{\widetilde{X}}(s)\right) e_j d\xi ds \\ &\quad + \sum_{k=1}^{\infty} \int_0^t \left(\operatorname{div}\left(\mu_k \widetilde{\widetilde{X}}(s)\right) e_k, e_j\right)_{-1} d\beta_k(s),\end{aligned}$$

for all $j \in \mathbb{N}$, where $\{e_j\}_j$ is the orthonormal basis considered above, and for all $t \in [0, T]$.

The martingale solution is a weak solution from the PDE and also from the stochastic point of view. For more details see [17] and see [11] for a similar approach.

The main result of this section is the following.

Theorem 3.2. *Under the assumptions (2) and* ***Hypotheses,*** *for each* $x \in L^{m+1}(\mathcal{O})$, *there is at least one martingale solution* $\left(\widetilde{\widetilde{\Omega}}, \widetilde{\widetilde{\mathcal{F}}}, \left(\widetilde{\widetilde{\mathcal{F}_t}}\right)_{t\geq 0}, \widetilde{\widetilde{\mathbb{P}}}, \widetilde{\widetilde{W}}, \widetilde{\widetilde{X}}\right)$ *to equation (1). Moreover, we have that*

$$\widetilde{\widetilde{X}} \in L^2\left(\widetilde{\widetilde{\Omega}}, L^\infty\left(0, T; H^{-1}(\mathcal{O})\right)\right) \cap L^{m+1}\left(\widetilde{\widetilde{\Omega}} \times (0, T) \times \mathcal{O}\right).$$

Proof. In order to approximate the equation with a mollifier as in the previous case, we shall first rewrite the operator B as

$$\begin{aligned}B(u) &= \sum_{k=1}^{\infty} \operatorname{div}(\mu_k u)(e_k, \cdot)_2 e_k \\ &= \sum_{k=1}^{\infty} \operatorname{div}(\mu_k \delta * u)(e_k, \cdot)_2 e_k\end{aligned}$$

where δ is the Dirac function and keeping in mind that $\delta * u = u$.

By taking a mollifier sequence $\{\delta_\varepsilon\}_\varepsilon$ which is defined as in the previous section, we can approximate the operator B as follows

$$\begin{aligned}B_\varepsilon &: H^{-1}(\mathcal{O}) \longrightarrow L_2\left(L^2(\mathcal{O}); H^{-1}(\mathcal{O})\right) \\ B_\varepsilon(u) &= \sum_{k=1}^{\infty} \operatorname{div}(\mu_k \delta_\varepsilon * u)(e_k, \cdot)_2 e_k, \quad u \in H^{-1}(\mathcal{O})\end{aligned}$$

where

$$\begin{aligned}\delta_\varepsilon * u &: \mathcal{O} \longrightarrow \mathbb{R} \\ (\delta_\varepsilon * u)(\xi) &= \int_{\mathcal{O}} u(x)\, \delta_\varepsilon(\xi - x)\, dx.\end{aligned}$$

We can easily check now that the approximating equation

$$\begin{cases} dX^{\varepsilon}(t) - \Delta\Psi(X^{\varepsilon}(t))\,dt = B_{\varepsilon}(X^{\varepsilon}(t))\,dW(t), & (0,T)\times\mathcal{O} \\ X^{\varepsilon}(t) = 0, & (0,T)\times\partial\mathcal{O} \\ X^{\varepsilon}(0) = x, & \mathcal{O} \end{cases} \tag{27}$$

has a unique solution X^{ε}.

Indeed, by arguing as in the previous section we have that B_{ε} is Lipschitz from $H^{-1}(\mathcal{O})$ into $L_2\left(L^2(\mathcal{O});H^{-1}(\mathcal{O})\right)$ for each ε fixed, i.e.

$$\begin{aligned} \|B_{\varepsilon}(u)\|^2_{L_2(L^2(\mathcal{O});H^{-1}(\mathcal{O}))} &= \sum_{k=1}^{\infty}|B_{\varepsilon}(u)e_k|^2_{-1} \\ &\leq \sum_{k=1}^{\infty}\left|\langle\mu_k,\nabla(\delta_{\varepsilon}*u)\rangle_{\mathbb{R}^d}e_k\right|^2_{-1} \\ &\leq C\sum_{k=1}^{\infty}\lambda_k^2|\mu_k|^2_{\mathbb{R}^d}|\delta_{\varepsilon}*u|^2_2 \\ &\leq C(\varepsilon)|u|^2_{-1}, \end{aligned}$$

and then we have a solution which satisfies $\mathbb{P}-a.s.$

$$\begin{aligned} (X_{\varepsilon}(t),e_j)_{-1} &= (x,e_j)_{-1} - \int_0^t\int_{\mathcal{O}}\Psi(X^{\varepsilon}(s))e_j\,d\xi ds \\ &\quad + \int_0^t\left((B_{\varepsilon}(X^{\varepsilon}(s))),e_j\right)_{-1}dW(s), \end{aligned}$$

$\forall j\in\mathbb{N}$ and $\forall t\in[0,T]$.

Note that by using Remark 3.1.4 from [7], the previous relation can be equivalently written as

$$X^{\varepsilon}(t) = x + \Delta\int_0^t\Psi(X^{\varepsilon}(s))\,ds + \int_0^t B_{\varepsilon}(X^{\varepsilon}(s))\,dW(s), \quad t\in[0,T].$$

By Itô's formula we obtain $\mathbb{P}-a.s.$ that

$$\begin{aligned} &\frac{1}{2}|X^{\varepsilon}(t)|^2_{-1} + \int_0^t\int_{\mathcal{O}}\Psi(X^{\varepsilon}(s))X^{\varepsilon}(s)\,d\xi ds \\ &= \frac{1}{2}|x|^2_{-1} + \int_0^t\|B_{\varepsilon}(X^{\varepsilon}(s))\|^2_{L_2(L^2(\mathcal{O});H^{-1}(\mathcal{O}))}ds + M_t, \end{aligned}$$

where

$$M_t = \int_0^t\langle X^{\varepsilon}(s),B_{\varepsilon}(X^{\varepsilon}(s))\,dW(s)\rangle_{-1},$$

is a continuous local martingale such that

$$\langle M\rangle_t = 2\int_0^t\left|(B_{\varepsilon}(X^{\varepsilon}(s)))^*X^{\varepsilon}(s)\right|^2_2ds, \quad t\geq 0,$$

and $(B_{\varepsilon}(X_{\varepsilon}(s)))^*$ is the adjoint of

$$B_{\varepsilon}(X_{\varepsilon}(s)) : L^2(\mathcal{O})\to H^{-1}(\mathcal{O}).$$

We can first easily check that

$$\int_0^t\|B_{\varepsilon}(X^{\varepsilon}(s))\|^2_{L_2(L^2(\mathcal{O});H^{-1}(\mathcal{O}))}ds \leq \overline{C}\int_0^t|X_{\varepsilon}(s)|^2_2\,ds,$$

where C is independent of ε.

Indeed, by arguing as in the first part of this work, we have that

$$\begin{aligned}\|B_\varepsilon(X^\varepsilon(s))\|^2_{L_2(L^2(\mathcal{O});H^{-1}(\mathcal{O}))} &\leq C_6^2\sum_{k=1}^\infty \lambda_k^2 |\mu_K|^2_{\mathbb{R}^d} |\delta_\varepsilon * X^\varepsilon(s)|_2^2 \\ &\leq C_6^2 C_0 |\delta_\varepsilon|^2_{L^1(\mathcal{O})} |X^\varepsilon(s)|_2^2,\end{aligned}$$

and since $\int_{\mathbb{R}^d}\delta_\varepsilon(y)\,dy=1$ we get that

$$\|B_\varepsilon(X^\varepsilon(s))\|^2_{L_2(L^2(\mathcal{O});H^{-1}(\mathcal{O}))}\leq \overline{C}\,|X^\varepsilon(s)|_2^2,$$

where the constant $\overline{C}$ is independent of ε.

By the Burkholder-Davis-Gundy inequality, we get for all $r\in[0,T]$ that

$$\begin{aligned}&\frac12\mathbb{E}\sup_{t\in[0,r]}|X^\varepsilon(t)|_{-1}^2+\mathbb{E}\int_0^r\int_{\mathcal{O}}\left(C_3|X^\varepsilon(s)|^{m+1}+C_4|X^\varepsilon(s)|^2\right)d\xi ds \quad (28)\\ \leq\ &\frac12|x|_{-1}^2+\overline{C}\mathbb{E}\int_0^r\int_{\mathcal{O}}|X^\varepsilon(s)|^2 d\xi ds\\ &+\overline{\overline{C}}\mathbb{E}\left(\int_0^r\left|(B_\varepsilon(X^\varepsilon(s)))^*X^\varepsilon(s)\right|_2^2ds\right)^{1/2},\end{aligned}$$

where $\overline{\overline{C}}$ is also independent of ε.

Keeping in mind that $B_\varepsilon(X_\varepsilon)$ is a Hilbert-Schmidt operator and therefore

$$\|B_\varepsilon(X^\varepsilon(s))\|^2_{L_2(L^2(\mathcal{O});H^{-1}(\mathcal{O}))}=\left\|(B_\varepsilon(X^\varepsilon(s)))^*\right\|^2_{L_2(H^{-1}(\mathcal{O});L^2(\mathcal{O}))},$$

we can compute

$$\begin{aligned}&\overline{\overline{C}}\mathbb{E}\left(\int_0^r\left|(B_\varepsilon(X^\varepsilon(s)))^*X^\varepsilon(s)\right|_2^2ds\right)^{1/2}\\ \leq\ &\overline{\overline{C}}\mathbb{E}\left(\int_0^r\|B_\varepsilon(X^\varepsilon(s))\|^2_{L_2(L^2(\mathcal{O});H^{-1}(\mathcal{O}))}|X^\varepsilon(s)|_{-1}^2ds\right)^{1/2}\\ \leq\ &\overline{\overline{C}}\mathbb{E}\left[\sup_{s\in[0,r]}|X^\varepsilon(s)|_{-1}\left(\int_0^r\|B_\varepsilon(X^\varepsilon(s))\|^2_{L_2(L^2(\mathcal{O});H^{-1}(\mathcal{O}))}ds\right)^{1/2}\right]\\ \leq\ &\overline{\overline{C}}\mathbb{E}\left[\sup_{s\in[0,r]}|X^\varepsilon(s)|_{-1}\left(\int_0^r\overline{C}\int_{\mathcal{O}}|X^\varepsilon(s)|^2d\xi ds\right)^{1/2}\right]\\ \leq\ &\frac14\mathbb{E}\left[\sup_{s\in[0,r]}|X^\varepsilon(s)|_{-1}^2\right]+\overline{\overline{C}}\overline{C}\mathbb{E}\int_0^r\int_{\mathcal{O}}|X^\varepsilon(s)|^2d\xi ds.\end{aligned}$$

By replacing the previous relation in(28) we get that

$$\begin{aligned}&\frac14\mathbb{E}\sup_{t\in[0,r]}|X^\varepsilon(t)|_{-1}^2\\ &+\mathbb{E}\int_0^r\int_{\mathcal{O}}\left(C_3|X^\varepsilon(s)|^{m+1}+\left(C_4-\overline{C}-\overline{\overline{C}}\overline{C}\right)|X^\varepsilon(s)|^2\right)d\xi ds\\ \leq\ &\frac12|x|_{-1}^2,\end{aligned}$$

and, since C_4 is assumed to be large enough, this leads to

$$\begin{aligned} & \mathbb{E} \sup_{t\in[0,r]} \left|X^{\varepsilon}(t)\right|_{-1}^{2} \\ & + \mathbb{E}\int_0^r \int_{\mathcal{O}} C_3 \left|X^{\varepsilon}(s)\right|^{m+1} d\xi ds + \mathbb{E}\int_0^r \int_{\mathcal{O}} \left|X^{\varepsilon}(s)\right|^{2} d\xi ds \\ \leq \ & C\left|x\right|_{-1}^{2}, \end{aligned}$$

for $\varepsilon > 0$, where C is a positive constant independent of ε.

Then, on a subsequence again denoted in the same way, we have the existence of

$$X \in L^{m+1}\left(\Omega \times (0,T) \times \mathcal{O}\right) \cap L^{2}\left(\Omega; C\left([0,T]; H^{-1}(\mathcal{O})\right)\right)$$

such that for $\varepsilon \to 0$

$$\begin{aligned} X^{\varepsilon} \ \rightharpoonup \ & X, \quad \text{weakly in } L^{m+1}\left(\Omega \times (0,T) \times \mathcal{O}\right) \\ & \text{and weak* in } L^{2}\left(\Omega; C\left([0,T]; H^{-1}(\mathcal{O})\right)\right) \end{aligned}$$

and

$$\eta \in L^{\frac{m+1}{m}}\left(\Omega \times (0,T) \times \mathcal{O}\right)$$

such that

$$\Psi\left(X^{\varepsilon}\right) \rightharpoonup \eta \text{ weakly in } L^{\frac{m+1}{m}}\left(\Omega \times (0,T) \times \mathcal{O}\right).$$

Since the weak convergences above are not sufficient to conclude the proof, we shall replace $\{X^{\varepsilon}\}$ by a sequence $\left\{\widetilde{X}^{\varepsilon}\right\}$ of processes defined in a probability space $\left(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}, \widetilde{W}\right)$ such that X^{ε} and $\widetilde{X}^{\varepsilon}$ have the same law.

To this purpose we consider the sequence of probability measures $\{\nu_{\varepsilon}\}_{\varepsilon}$, where ν_{ε} is the law of X^{ε}, and we prove that $\{\nu_{\varepsilon}\}_{\varepsilon}$ is tight in the space $C\left([0,T]; H^{-1}(\mathcal{O})\right)$. We recall that this means that, for each $\delta > 0$ there is a compact subset B of $C\left([0,T]; H^{-1}(\mathcal{O})\right)$ such that $\nu_{\varepsilon}\left(B^{c}\right) \leq \delta$ for all $\varepsilon > 0$.

We define for each $r > 0$ and $\gamma > 0$, the set

$$\begin{aligned} B_{r,\gamma} \ = \ & \Big\{ y \in C\left([0,T]; H^{-1}(\mathcal{O})\right) : \sup_{t\in[0,T]} \left|y(t)\right|_{-1} \leq r, \ \|y\|_{L^{\infty}(0,T.L^{2}(\mathcal{O}))} \leq r \\ & \quad \textit{and } \left|y(t) - y(s)\right|_{-1} \leq \gamma\left|t-s\right|^{1/2}, \ \forall t,s \in [0,T] \Big\}. \end{aligned}$$

Since the set is uniformly bounded and satisfies a Hölder condition of order $1/2$ with a fixed constant γ, we have by Arzelà-Ascoli theorem that $B_{r,\gamma}$ is compact in $C\left([0,T]; H^{-1}(\mathcal{O})\right)$.

We can apply the Itô formula to equation (27) with the Lyapunov function

$$\varphi(u) = \int_{\mathcal{O}} \left|u(\xi)\right|^{2} d\xi, \quad \forall u \in L^{2}(\mathcal{O}).$$

In fact we apply the Itô formula with

$$\varphi_{\nu}(u) = \varphi\left(\left(Id - \nu\Delta\right)^{-1} u\right) \overset{Note}{=} \varphi\left(J_{\nu}(u)\right)$$

where $J_\nu = (Id - \nu\Delta)^{-1}$ is the resolvent of the Laplace operator, and we get that

$$\begin{aligned} & \mathbb{E}\left|J_\nu X^\varepsilon(s)\right|_2^2 + \mathbb{E}\int_0^s \int_{\mathcal{O}} J_\nu X^\varepsilon(\theta)\, J_\nu A\left(X^\varepsilon(\theta)\right) d\xi d\theta \\ \leq\ & \left|J_\nu x\right|_2^2 + \frac{1}{2}\mathbb{E}\int_0^s \sum_{k=1}^\infty \left|J_\nu B_\varepsilon\left(X^\varepsilon(\theta)\right) e_k\right|_2^2 d\theta. \end{aligned}$$

Keeping in mind that the resolvent of the Laplace operator is strongly convergent in $L^2(\Omega \times (0,T) \times \mathcal{O})$ (see e.g. [2]) we can pass to the limit for $\nu \to 0$ as in [4], [9], [13] or [14].

We obtain

$$\begin{aligned} & \mathbb{E}\left|X^\varepsilon(s)\right|_2^2 + \mathbb{E}\int_0^s \int_{\mathcal{O}} \left|\nabla X^\varepsilon(\theta)\right|^2 \Psi'\left(X^\varepsilon(\theta)\right) d\xi d\theta \\ \leq\ & |x|_2^2 + \frac{1}{2}\mathbb{E}\int_0^s \sum_{k=1}^\infty \lambda_k^2 \left|\mu_k\right|_{\mathbb{R}^d}^2 \int_{\mathcal{O}} \left|\nabla X^\varepsilon(\theta)\right|^2 d\xi d\theta, \end{aligned}$$

and then, by using the strong monotonicity property of Ψ, we get that

$$\mathbb{E}\left|X^\varepsilon(s)\right|_2^2 + \left(C_5 - \frac{C_0}{2}\right)\mathbb{E}\int_0^s \int_{\mathcal{O}} \left|\nabla X^\varepsilon(\theta)\right|^2 d\xi d\theta \leq C.$$

From the previous relation we get also that

$$\mathbb{E}\int_0^s \int_{\mathcal{O}} \left|\nabla X^\varepsilon(\theta)\right|^2 \Psi'\left(X^\varepsilon(\theta)\right) d\xi d\theta \leq C.$$

We shall apply Itô's formula to (27) with the Lyapunov function

$$\varphi(u) = \begin{cases} \int_{\mathcal{O}} j\left(u(\xi)\right) d\xi, & u \in L^1(u),\ j(u) \in L^1(u) \\ \infty, & if\ not \end{cases}$$

and we get

$$\begin{aligned} & \mathbb{E}\int_{\mathcal{O}} j\left(X^\varepsilon\right) d\xi + \mathbb{E}\int_0^t \int_{\mathcal{O}} \left|\nabla \Psi\left(X^\varepsilon\right)\right|^2 d\xi d\theta \\ \leq\ & \mathbb{E}\int_{\mathcal{O}} j(x)\, d\xi + C\mathbb{E}\int_0^s \int_{\mathcal{O}} \left|\nabla X^\varepsilon(\theta)\right|^2 \Psi'\left(X^\varepsilon(\theta)\right) d\xi d\theta \leq C \end{aligned}$$

and then

$$\mathbb{E}\int_{\mathcal{O}} j\left(X^\varepsilon\right) d\xi \leq C. \tag{29}$$

Finally, by applying again the Itô formula to the process

$$t \longmapsto \left|X^\varepsilon(t) - X^\varepsilon(s)\right|_{-1}^2$$

we get that

$$\begin{aligned} & \frac{1}{2}\mathbb{E}\left|X^\varepsilon(t) - X^\varepsilon(s)\right|_{-1}^2 + \mathbb{E}\int_s^t \int_{\mathcal{O}} \Psi\left(X^\varepsilon(\theta)\right)\left(X^\varepsilon(\theta) - X^\varepsilon(s)\right) d\xi d\theta \\ \leq\ & \widetilde{C}\mathbb{E}\int_s^t \int_{\mathcal{O}} \left|X^\varepsilon(\theta)\right|^2 d\xi d\theta. \end{aligned}$$

Since

$$\begin{aligned} & \mathbb{E}\int_s^t \int_{\mathcal{O}} \Psi\left(X^\varepsilon(\theta)\right)\left(X^\varepsilon(\theta) - X^\varepsilon(s)\right) d\xi d\theta \\ \geq\ & \mathbb{E}\int_s^t \int_{\mathcal{O}} j\left(X^\varepsilon(\theta)\right) d\xi d\theta - \mathbb{E}\int_s^t \int_{\mathcal{O}} j\left(X^\varepsilon(s)\right) d\xi d\theta \end{aligned}$$

$$\geq \quad C_4\mathbb{E}\int_s^t\int_{\mathcal{O}}|X^\varepsilon(\theta)|^2\,d\xi d\theta-\mathbb{E}(t-s)\int_{\mathcal{O}}j(X^\varepsilon(s))\,d\xi.$$

We get by (29) that

$$\begin{aligned}&\frac{1}{2}\mathbb{E}\left|X^\varepsilon(t)-X^\varepsilon(s)\right|_{-1}^2+\left(C_4-\widetilde{C}\right)\mathbb{E}\int_s^t\int_{\mathcal{O}}|X^\varepsilon(\theta)|^2\,d\xi d\theta\\ \leq\quad&(t-s)\,\mathbb{E}\int_{\mathcal{O}}j(X^\varepsilon(s))\,d\xi\leq C(t-s).\end{aligned}$$

We obtain that

$$\mathbb{E}\left|X^\varepsilon(t)-X^\varepsilon(s)\right|_{-1}^2\leq 2C(t-s).$$

Finally, by using the Tchebychev inequality

$$\mathbb{P}\left[|X^\varepsilon|_{-1}\geq r\right]\leq\frac{1}{r}\mathbb{E}\left|X^\varepsilon\right|_{-1}$$

we can conclude that for each δ there exist γ and r, independent of ε such that

$$\nu_\varepsilon\left(B_{r,\gamma}^c\right)=\mathbb{P}\left(X^\varepsilon\in B_{r,\gamma}^c\right)\leq\frac{1}{r}\mathbb{E}\sup_{t\in[0,T]}\left|X^\varepsilon(t)\right|_{-1}^2<\delta,$$

and therefore $\{\nu_\varepsilon\}_{\varepsilon>0}$ is tight.

Then, by the Skorohod theorem, we have a probability space $\left(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathbb{P}}\right)$ and the stochastic process $\widetilde{X}$ and $\left(\widetilde{X}^\varepsilon\right)_{\varepsilon>0}$ on $\left(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathbb{P}}\right)$ such that the law of $\widetilde{X}^\varepsilon$ is the same as the law of X^ε and

$$\widetilde{X}^\varepsilon\longrightarrow\widetilde{X}\text{ in }C\left([0,T];H^{-1}(\mathcal{O})\right),\quad\mathbb{P}-a.s.\tag{30}$$

as $\varepsilon\longrightarrow 0$. We have also that the law of $\widetilde{X}$ is the same as the law of X.

Since

$$\int_0^t\left|\widetilde{X}^\varepsilon-\widetilde{X}\right|_{-1}^2ds\to 0,\quad\mathbb{P}-a.s.$$

we can use the Egorov theorem to get that

$$\mathbb{E}\int_0^t\left|\widetilde{X}^\varepsilon-\widetilde{X}\right|_{-1}^2ds\to 0.$$

Indeed, we have for $\forall\ \delta>0$ a subset A_δ of $\widetilde{\Omega}$ such that $\mathbb{P}\left(\widetilde{\Omega}\backslash A_\delta\right)<\delta$ and

$$\int_0^t\left|\widetilde{X}^\varepsilon-\widetilde{X}\right|_{-1}^2ds\to 0,\quad\text{uniformly on }A_\delta.$$

We see by using the Hölder inequality and since $m>1$ that

$$\begin{aligned}&\int_{\widetilde{\Omega}\backslash A_\delta}\int_0^t\left|\widetilde{X}^\varepsilon-\widetilde{X}\right|_{-1}^2ds\,d\mathbb{P}\\ \leq\quad&\left(\int_{\widetilde{\Omega}\backslash A_\delta}\left(\int_0^t\left|\widetilde{X}^\varepsilon-\widetilde{X}\right|_{-1}^2ds\right)^{\frac{m+1}{2}}d\mathbb{P}\right)^{\frac{2}{m+1}}\left(\int_{\widetilde{\Omega}\backslash A_\delta}1d\mathbb{P}\right)^{\frac{m-1}{m+1}}\\ \leq\quad&\left(\mathbb{E}\int_0^t\left|\widetilde{X}^\varepsilon-\widetilde{X}\right|_{m+1}^{m+1}ds\right)^{\frac{2}{m+1}}\delta^{\frac{m-1}{m+1}}\leq C\delta^{\frac{m-1}{m+1}}.\end{aligned}$$

On the other hand, since we have the uniform convergence on A_δ, we get that for each δ there is $\varepsilon > 0$ such that

$$\int_{A_\delta} \left(\int_0^t \left| \widetilde{X}^\varepsilon - \widetilde{X} \right|_{-1}^2 ds \right) d\mathbb{P} \leq \delta.$$

We conclude that

$$\widetilde{X}^\varepsilon \longrightarrow \widetilde{X} \text{ strongly in } L^2 \left(\widetilde{\Omega}; L^2 \left([0,T] ; H^{-1} (\mathcal{O}) \right) \right).$$

Since the law of $\widetilde{X}^\varepsilon$ is the same as the law of X^ε and keeping in mind that

$$\mathbb{E} \int_0^t \int_{\mathcal{O}} \left| \nabla \Psi \left(\widetilde{X}^\varepsilon \right) \right|^2 d\xi ds < C$$

we obtain that

$$\Psi \left(\widetilde{X}^\varepsilon \right) \to \Psi \left(\widetilde{X} \right) \text{ weakly in } L^2 \left(\widetilde{\Omega}; L^2 \left([0,T] ; H_0^1 (\mathcal{O}) \right) \right).$$

We shall show now that, for each ε fixed, the process

$$\widetilde{M}_\varepsilon (t) = \widetilde{X}^\varepsilon (t) - x - \Delta \int_0^t \Psi \left(\widetilde{X}^\varepsilon (s) \right) ds, \quad [0,T],$$

is a square integrable martingale with respect to

$$\widetilde{\mathcal{F}}_\varepsilon (t) = \sigma \left\{ \widetilde{X}^\varepsilon (s), \quad s \leq t \right\}, \quad [0,T],$$

the filtration generated by $\left\{ \widetilde{X}^\varepsilon (t) \right\}_{t \in [0,T]}$ and that the quadratic variation of $\widetilde{M}_\varepsilon$ is

$$\left\langle \widetilde{M}_\varepsilon \right\rangle_t = \int_0^t B_\varepsilon \left(\widetilde{X}^\varepsilon (s) \right) \left(B_\varepsilon \left(\widetilde{X}^\varepsilon (s) \right) \right)^* ds$$

where $\left(B_\varepsilon \left(\widetilde{X}^\varepsilon (s) \right) \right)^*$ is the adjoint of $B_\varepsilon \left(\widetilde{X}^\varepsilon (s) \right)$.

All this is true for the process

$$M_\varepsilon (t) = X^\varepsilon (t) - x - \Delta \int_0^t \Psi \left(X^\varepsilon (s) \right) ds$$

because

$$M_\varepsilon (t) = \int_0^t B_\varepsilon \left(X^\varepsilon (s) \right) dW_s, \quad t \in [0,T].$$

Since X^ε and $\widetilde{X}^\varepsilon$ have the same law, we have the previous properties also for $\widetilde{M}_\varepsilon$. More precisely we get first that $\mathbb{E} \left(\left| \widetilde{M}_\varepsilon (t) \right|_{-1}^2 \right) < \infty$. Then $\widetilde{M}_\varepsilon$ is a $\sigma \left\{ \widetilde{X}^\varepsilon (s) | 0 \leq s \leq t \right\}$, $t \in [0,T]$ martingale, since

$$\mathbb{E} \left(\left[\widetilde{M}_\varepsilon (t) - \widetilde{M}_\varepsilon (s) \right] \varphi \left(\widetilde{X}^\varepsilon (\cdot) \right) \right) = 0, \tag{31}$$

for $0 \leq s \leq t \leq T$ and for all φ who is a real valued, bounded and continuous functions on $C \left([0,T] ; H^{-1} (\mathcal{O}) \right)$.

We have also that

$$
\begin{aligned}
& \mathbb{E}\left(\left[\left\langle \widetilde{M}_\varepsilon(t), a\right\rangle_{-1}\left\langle \widetilde{M}_\varepsilon(t), b\right\rangle_{-1}\right.\right. && (32)\\
& \left.\left.-\int_0^t\left\langle B_\varepsilon\left(\widetilde{X}^\varepsilon(s)\right)\left(B_\varepsilon\left(\widetilde{X}^\varepsilon(s)\right)\right)^* a, b\right\rangle_{-1} ds\right]\varphi\left(\widetilde{X}^\varepsilon(\cdot)\right)\right)\\
= \;& 0,
\end{aligned}
$$

for all $a, b \in H^{-1}(\mathcal{O})$, and therefore we have the quadratic variation of $\widetilde{M}_\varepsilon$. (For more details see the similar proof from [17], page 232.)

We shall now check that we can take the limit as $\varepsilon \longrightarrow 0$ in the previous relations and then we will get that the process

$$
\widetilde{M}(t) = \widetilde{X}(t) - x - \Delta\int_0^t \Psi\left(\widetilde{X}(s)\right) ds, \quad t \in [0, T],
$$

is a $H^{-1}(\mathcal{O})$ valued martingale with respect to the filtration

$$
\widetilde{\mathcal{F}}(t) = \sigma\left\{\widetilde{X}(s), \quad s \le t\right\}, \quad t \in [0, T],
$$

having the quadratic variation

$$
\left\langle \widetilde{M}\right\rangle_t = \int_0^t B\left(\widetilde{X}(s)\right)\left(B\left(\widetilde{X}(s)\right)\right)^* ds.
$$

To this purpose, by the same argument as in [17], page 232, we see that

$$
\sup_\varepsilon \mathbb{E}\left(\left|\widetilde{M}_\varepsilon(t)\right|_{-1}^2\right) = \sup_\varepsilon \mathbb{E}\left(\left|M_\varepsilon(t)\right|_{-1}^2\right)
$$

and therefore the sequence $\left\{\widetilde{M}_\varepsilon\right\}_\varepsilon$ is uniformly integrable

$$
\lim_{\varepsilon\longrightarrow 0}\mathbb{E}\left(\left|\widetilde{M}_\varepsilon(t)\right|_{-1}^2\right) = \mathbb{E}\left|\widetilde{M}(t)\right|_{-1}^2 < \infty, \quad t \in [0, T].
$$

Consequently $\widetilde{M}$ is a square integrable process.

Following again the idea from [17] we shall continue by defining the martingales

$$
\begin{aligned}
\widetilde{N}_\varepsilon(t) &= \Delta^{-1}\widetilde{M}_\varepsilon(t)\\
&= \Delta^{-1}\widetilde{X}^\varepsilon(t) - \Delta^{-1}x - \int_0^t \Psi\left(\widetilde{X}^\varepsilon(s)\right) ds, \quad t \in [0, T].
\end{aligned}
$$

By the same argument as above, $\widetilde{N}_\varepsilon$ is a square integrable continuous martingale on $\left(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}\right)$ with the quadratic variation

$$
\left\langle \widetilde{N}_\varepsilon\right\rangle_t = \int_0^t \left[\Delta^{-1}B_\varepsilon\left(\widetilde{X}^\varepsilon(s)\right)\right]\left[\Delta^{-1}\left(B_\varepsilon\left(\widetilde{X}^\varepsilon(s)\right)\right)\right]^* ds.
$$

To this purpose, it is sufficient now to pass to the limit in

$$
\mathbb{E}\left(\left[\widetilde{N}_\varepsilon(t) - \widetilde{N}_\varepsilon(s)\right]\varphi\left(\widetilde{X}^\varepsilon(\cdot)\right)\right) = 0, \quad 0 \le s \le t \le T, \tag{33}
$$

and in

$$
\mathbb{E}\left(\left[\left\langle \widetilde{N}_\varepsilon(t), a\right\rangle_{-1}\left\langle \widetilde{N}_\varepsilon(t), b\right\rangle_{-1}\right.\right. \tag{34}
$$

$$
\begin{aligned}
& -\int_0^t \left\langle \Delta^{-1} B_\varepsilon\left(\widetilde{X}^\varepsilon(s)\right)\left(\Delta^{-1} B_\varepsilon\left(\widetilde{X}^\varepsilon(s)\right)\right)^* a, b\right\rangle_{-1} ds\Big] \varphi\left(\widetilde{X}^\varepsilon(\cdot)\right)\Big) \\
= \;& 0.
\end{aligned}
$$

We shall now pass to the limit in

$$
\begin{aligned}
& \mathbb{E}\left(\int_0^t \left\langle \Delta^{-1} B_\varepsilon\left(\widetilde{X}^\varepsilon(s)\right)\left(\Delta^{-1} B_\varepsilon\left(\widetilde{X}^\varepsilon(s)\right)\right)^* a, b\right\rangle_{-1} ds\right) \\
= \;& \mathbb{E}\left(\int_0^t \left\langle \left(\Delta^{-1} B_\varepsilon\left(\widetilde{X}^\varepsilon(s)\right)\right)^* a, \left(\Delta^{-1} B_\varepsilon\left(\widetilde{X}^\varepsilon(s)\right)\right)^* b\right\rangle_2 ds\right)
\end{aligned}
$$

To this purpose we compute

$$
\begin{aligned}
& \mathbb{E}\int_0^t \left(\left\langle \left(\Delta^{-1} B_\varepsilon\left(\widetilde{X}^\varepsilon(s)\right)\right)^* a, \left(\Delta^{-1} B_\varepsilon\left(\widetilde{X}^\varepsilon(s)\right)\right)^* b\right\rangle_2 \right. \\
& \left. -\left\langle \left(\Delta^{-1} B\left(\widetilde{X}(s)\right)\right)^* a, \left(\Delta^{-1} B\left(\widetilde{X}(s)\right)\right)^* b\right\rangle_2\right) ds \\
\leq \;& \mathbb{E}\int_0^t \left(\left\langle \left(\Delta^{-1} B_\varepsilon\left(\widetilde{X}^\varepsilon(s)\right)\right)^* a, \left(\Delta^{-1}\left(B_\varepsilon\left(\widetilde{X}^\varepsilon(s)\right) - B\left(\widetilde{X}(s)\right)\right)\right)^* b\right\rangle_2 \right. \\
& \left. +\left\langle \left(\Delta^{-1}\left(B_\varepsilon\left(\widetilde{X}^\varepsilon(s)\right) - B\left(\widetilde{X}(s)\right)\right)\right)^* a, \left(\Delta^{-1} B\left(\widetilde{X}(s)\right)\right)^* b\right\rangle_2\right) ds \\
\leq \;& \mathbb{E}\int_0^t \left(\left|\Delta^{-1} B_\varepsilon\left(\widetilde{X}^\varepsilon(s)\right)\right|_{L_2(L^2(\mathcal{O});H^{-1}(\mathcal{O}))} |a|_{-1} \right. \\
& \left. \left|\Delta^{-1}\left(B_\varepsilon\left(\widetilde{X}^\varepsilon(s)\right) - B\left(\widetilde{X}(s)\right)\right)\right|_{L_2(L^2(\mathcal{O});H^{-1}(\mathcal{O}))} |b|_{-1}\right) ds \\
& +\mathbb{E}\int_0^t \left(\left|\Delta^{-1} B\left(\widetilde{X}(s)\right)\right|_{L_2(L^2(\mathcal{O});H^{-1}(\mathcal{O}))} |b|_{-1} \right. \\
& \left. \left|\Delta^{-1}\left(B_\varepsilon\left(\widetilde{X}^\varepsilon(s)\right) - B\left(\widetilde{X}(s)\right)\right)\right|_{L_2(L^2(\mathcal{O});H^{-1}(\mathcal{O}))} |a|_{-1}\right) ds \\
\leq \;& C\left(\mathbb{E}\int_0^t \left|\Delta^{-1}\left(B_\varepsilon\left(\widetilde{X}^\varepsilon(s)\right) - B\left(\widetilde{X}(s)\right)\right)\right|^2_{L_2(L^2(\mathcal{O});H^{-1}(\mathcal{O}))} ds\right)^{1/2}.
\end{aligned}
$$

It is then sufficient to pass to the limit in

$$
\mathbb{E}\int_0^t \left|\Delta^{-1}\left(B_\varepsilon\left(\widetilde{X}^\varepsilon(s)\right) - B\left(\widetilde{X}(s)\right)\right)\right|^2_{L_2(L^2(\mathcal{O});H^{-1}(\mathcal{O}))} ds. \tag{35}
$$

We shall first see that

$$
\begin{aligned}
& \mathbb{E}\int_0^t \left|\Delta^{-1}\left(B_\varepsilon\left(\widetilde{X}^\varepsilon(s)\right) - B\left(\widetilde{X}(s)\right)\right)\right|^2_{L_2(L^2(\mathcal{O});H^{-1}(\mathcal{O}))} ds \\
= \;& \mathbb{E}\int_0^t \sum_{k=1}^\infty \left|\Delta^{-1}\left\langle \mu_k, \nabla\left(\delta_\varepsilon * \widetilde{X}^\varepsilon(s) - \widetilde{X}(s)\right)\right\rangle_{\mathbb{R}^d} e_k\right|^2_{-1} ds \\
\leq \;& \mathbb{E}\int_0^t \sum_{k=1}^\infty |\mu_k|^2_{\mathbb{R}^d} \lambda_k^2 \left|\delta_\varepsilon * \widetilde{X}^\varepsilon(s) - \widetilde{X}(s)\right|^2_{-1} ds \\
\leq \;& C\mathbb{E}\int_0^t \left(\left|\delta_\varepsilon * \widetilde{X}^\varepsilon(s) - \widetilde{X}^\varepsilon(s)\right|^2_{-1} + \left|\widetilde{X}^\varepsilon(s) - \widetilde{X}(s)\right|^2_{-1}\right) ds \\
\stackrel{Note}{=} \;& C\left(T_1 + T_2\right).
\end{aligned}
$$

We can easily pass to the limit in T_2 by (30) and get that

$$T_2 = \mathbb{E}\int_0^t \left|\widetilde{X}^{\varepsilon}(s) - \widetilde{X}(s)\right|_{-1}^2 ds \longrightarrow 0 \text{ for } \varepsilon \to 0.$$

On the other hand, for T_1 we have that

$$\begin{aligned}
T_1 &= \mathbb{E}\int_0^t \left(\sup_{\varphi\in H_0^1(\mathcal{O}),\ \|\varphi\|_{H_0^1(\mathcal{O})}\le 1} \left| \int_{\mathcal{O}} \left(\delta_\varepsilon * \widetilde{X}^{\varepsilon}(s) - \widetilde{X}^{\varepsilon}(s)\right)\varphi d\xi \right| \right)^2 ds \\
&= \mathbb{E}\int_0^t \left(\sup_{\varphi\in H_0^1(\mathcal{O}),\ \|\varphi\|_{H_0^1(\mathcal{O})}\le 1} \left| \int_{\mathcal{O}} \widetilde{X}^{\varepsilon}(s)\left(\delta_\varepsilon * \varphi - \varphi\right) d\xi \right| \right)^2 ds \\
&\le \mathbb{E}\int_0^t \left(\sup_{\varphi\in H_0^1(\mathcal{O}),\ \|\varphi\|_{H_0^1(\mathcal{O})}\le 1} \left|\widetilde{X}^{\varepsilon}(s)\right|_{-1} \left|\delta_\varepsilon * \varphi - \varphi\right|_{H_0^1(\mathcal{O})} \right)^2 ds \\
&\le \mathbb{E}\int_0^t \left|\widetilde{X}^{\varepsilon}(s)\right|_{-1}^2 ds \left(\sup_{\varphi\in H_0^1(\mathcal{O}),\ \|\varphi\|_{H_0^1(\mathcal{O})}\le 1} \left|\delta_\varepsilon * \varphi - \varphi\right|_{H_0^1(\mathcal{O})} \right)^2 .
\end{aligned}$$

Since

$$\nabla\left(\delta_\varepsilon * \varphi\right) = \delta_\varepsilon * \nabla\varphi \longrightarrow \nabla\varphi \text{ for } \varepsilon \longrightarrow 0$$

and keeping in mind that $\mathbb{E}\int_0^t \left|\widetilde{X}^{\varepsilon}(s)\right|_{-1}^2 ds$ is bounded, we have that $T_1 \to 0$ for $\varepsilon \longrightarrow 0$. We can now pass to the limit in (35).

After passing to the limit in (33) and (34) we get that the process

$$\widetilde{N}(t) = \Delta^{-1}\widetilde{X}(t) - \Delta^{-1}x - \int_0^t \Psi\left(\widetilde{X}(s)\right) ds, \quad t \in [0,T],$$

is a square integrable martingale with respect to

$$\widetilde{\mathcal{F}}(t) = \sigma\left\{\widetilde{X}(s), \quad s \le t\right\}, \quad t \in [0,T],$$

for which

$$\left\langle \widetilde{N} \right\rangle_t = \int_0^t \left[\Delta^{-1}B\left(\widetilde{X}(s)\right)\right]\left[\Delta^{-1}\left(B\left(\widetilde{X}(s)\right)\right)\right]^* ds, \qquad t \in [0,T].$$

By the representation theorem (see e.g. [17] Theorem 8.2) we have the existence of a probability space $\left(\widetilde{\widetilde{\Omega}}, \widetilde{\widetilde{\mathcal{F}}}, \widetilde{\widetilde{\mathbb{P}}}\right)$, a filtration $\left\{\widetilde{\widetilde{\mathcal{F}}}\right\}$ a Wiener process $\widetilde{\widetilde{W}}$ and a predictable continuous process $\widetilde{\widetilde{X}}$ such that

$$\widetilde{\widetilde{X}}(t) = x + \Delta\int_0^t \Psi\left(\widetilde{\widetilde{X}}(s)\right) ds + \int_0^t B\left(\widetilde{\widetilde{X}}(s)\right) d\widetilde{\widetilde{W}}(s), \quad t \in [0,T].$$

The proof of the existence of at least one martingale solution is now complete. □

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